The structure of graphs with a vital linkage of order 2*

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Abstract

A *linkage of order* k of a graph G is a subgraph with k components, each of which is a path. A linkage is *vital* if it spans all vertices, and no other linkage connects the same pairs of end vertices. We give a characterization of the graphs with a vital linkage of order 2: they are certain minors of a family of highly structured graphs.

1 Introduction

Robertson and Seymour [4] defined a *linkage* in a graph G as a subgraph in which each component is a path. The *order* of a linkage is the number of components. A linkage L of order k is *unique* if no other collection of paths connects the same pairs of vertices, it is *spanning* if $V(L) = V(G)$, and it is *vital* if it is both unique and spanning. Graphs with a vital linkage are well-behaved. For instance, Robertson and Seymour proved the following:

Theorem 1.1 (Robertson and Seymour [4, Theorem 1.1]). *There exists an integer w, depending only on k, such that every graph with a vital linkage of order k has tree width at most w.*

Note that Robertson and Seymour use the term p-linkage to denote a linkage with p terminals. Robertson and Seymour's proof of this theorem is surprisingly elaborate, and uses their structural description of graphs with no large clique-minor. Recently Kawarabayashi and Wollan

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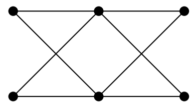

Figure 1: *The graph $K_{2,4}$.*

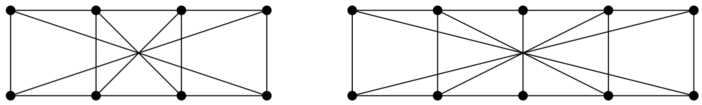

Figure 2: *The graphs $\ddot{U}_4$ and $\ddot{U}_5$.*

[2] proved a strengthening of this result. Their shorter proof avoids using the structure theorem.

Our interest in linkages, in particular those of order 2, stems from quite a different area of research: matroid theory. Truemper [5] studied a class of binary matroids that he calls *almost regular*. His proofs lean heavily on a class of matroids that are single-element extensions of the cycle matroids of graphs with a vital linkage of order 2. These matroids turned up again in the excluded-minor characterization of matroids that are either binary or ternary, by Mayhew et al. [3].

Truemper proves that an almost regular matroid can be built from one of two specific matroids by certain $\Delta - Y$ operations. This is a deep result, but it does not yield bounds on the branch width of these matroids. In a forthcoming paper the authors of this paper, together with Chun, will give an explicit structural description of the class of almost regular matroids [1]. The main result of this paper will be of use in that project.

To state our main result we need a few more definitions. Fix a graph G and a spanning linkage L of order k. A *path edge* is a member of $E(L)$; an edge in $E(G) - E(L)$ is called a *chord* if its endpoints lie in a single path, and a *rung edge* otherwise. If L is vital, then G cannot have any chords.

A *linkage minor* of G with respect to a (chordless) linkage L is a minor H of G such that all path edges in $E(G)-E(H)$ have been contracted, and all rung edges in $E(G) - E(H)$ have been deleted. If the linkage L is clear from the context we simply say that H is a linkage minor of G. Moreover, let G be a graph with a chordless 2-linkage L. If G has a linkage minor isomorphic to $K_{2,4}$, such that the terminals of L are mapped to the degree-2 vertices of $K_{2,4}$, we say that G has an *XX* linkage minor (cf. Figure 1).

For each integer n, the graph $\ddot{U}_n$ is the graph with $V(\ddot{U}_n) = \{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\}$, and

$$E(\ddot{U}_n) = \{v_i v_{i+1} \mid i = 1, \dots, n-1\} \cup \{u_i u_{i+1} \mid i = 1, \dots, n-1\} \cup \{u_i v_i \mid i = 1, \dots, n\} \cup \{u_i v_{n+1-i} \mid i = 1, \dots, n\}. \quad (1)$$

We denote by L_n the linkage of $\ddot{U}_n$ consisting of all edges $v_i v_{i+1}$ and $u_i u_{i+1}$ for $i = 1, \dots, n-1$. In Figure 2 the graphs $\ddot{U}_4$ and $\ddot{U}_5$ are depicted.

Finally, we say that G is a *Truemper graph* if G is a linkage minor of $\ddot{U}_n$ for some n. The main result of this paper is the following:

Theorem 1.2. *Let G be a graph. The following statements are equivalent:*

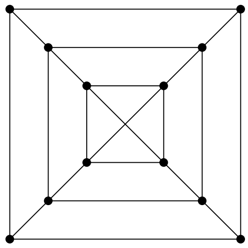

Figure 3: *The graph $\ddot{U}_6$. The linkage is formed by the two diagonally drawn paths.*

(i) *G has a vital linkage of order 2;*

(ii) *G has a chordless spanning linkage of order 2 with no XX linkage minor;*

(iii) *G is a Truemper graph.*

Robertson and Seymour [4] commented, without proof, that graphs with a vital linkage with $k \leq 5$ terminal vertices have path width at most k. A weaker claim is the following:

Corollary 1.3. *Let G be a graph with a vital linkage of order* 2*. Then G has path width at most 4.*

Another consequence of our result is that graphs with a vital linkage of order 2 embed in the projective plane:

Corollary 1.4. *Let G be a graph with a vital linkage of order 2. Then G can be embedded on a Möbius strip.*

Both corollaries can be seen to be true by considering an alternative depiction of $\ddot{U}_{2n}$, analogous to Figure 3.

2 Proof of Theorem 1.2

We start with a few more definitions. Suppose L is a linkage of order 2 with components P_1 and P_2, such that the terminal vertices of P_1 are s_1 and t_1, and those of P_2 are s_2 and t_2. We order the vertices on the paths in a natural way, as follows. If v and w are vertices of P_i, then we say that v is *(strictly) to the left* of w if the graph distance from s_i to v in the subgraph P_i is (strictly) smaller than the graph distance from s_i to w. The notion *to the right* is defined analogously.

We will frequently use the following elementary observation, whose proof we omit.

Lemma 2.1. *Let G be a graph with a chordless spanning linkage L of order 2. Let P_1 and P_2 be the components of L, with terminal vertices respectively s_1, t_1 and s_2, t_2. Let H be a linkage minor of G. If v and w are on P_i, and v is to the left of w, then the vertex corresponding to v in H is to the left of the vertex corresponding to w in H.*

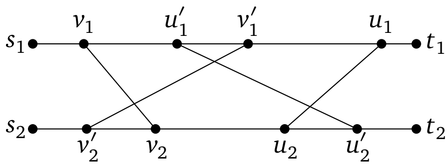

Figure 4: *Detail of the proof of Lemma 2.2.*

Without further ado we dive into the proof, which will consist of a sequence of lemmas. The first deals with the equivalence of the first two statements in the theorem.

Lemma 2.2. *Let G be a graph with a chordless spanning linkage L of order 2. Then L is vital if and only if G has no XX linkage minor with respect to L.*

Proof. First we suppose that there exists a graph G with a non-vital chordless spanning linkage L of order 2 such that G has no *XX* linkage minor. Let P_1, P_2 be the paths of L, where P_1 runs from s_1 to t_1, and P_2 runs from s_2 to t_2. Let P_1', P_2' be different paths connecting the same pairs of vertices. Without loss of generality, $P_1' \neq P_1$. But then P_1' must meet P_2, so $P_2' \neq P_2$. Let $e = v_1v_2$ be an edge of P_1' such that the subpath $s_1 - v_1$ of P_1' is also a subpath of P_1, but e is not an edge of P_1. Let $f = u_2u_1$ be an edge of P_1' such that the subpath $u_1 - t_1$ of P_1' is also a subpath of P_2, but f is not an edge of P_2. Similarly, let $e' = v_2'v_1'$ be an edge of P_2' such that the subpath $s_2 - v_2'$ of P_2' is also a subpath of P_2, but e' is not an edge of P_2. Let $f' = u_1'u_2'$ be an edge of P_2' such that the subpath $u_2' - t_2$ of P_2' is also a subpath of P_2, but f' is not on P_2. See Figure 4.

Since P_1' and P_2' are vertex-disjoint, v_2' must be strictly to the left of v_2 and u_2. For the same reason, v_1' must be strictly between v_1 and u_1. Likewise, u_2' must be strictly to the right of v_2 and u_2, and u_1' must be strictly between v_1 and u_1. Now construct a linkage minor H of G, as follows. Contract all edges on the subpaths $s_1 - v_1$, $v_1' - u_1'$, and $u_1 - t_1$ of P_1, contract all edges on the subpaths $s_2 - v_2'$, $v_2 - u_2$, and $u_2' - t_2$ of P_2, delete all rung edges but $\{e, f, e', f'\}$, and contract all but one of the edges of each series class in the resulting graph. Clearly H is isomorphic to *XX*, a contradiction.

Conversely, suppose that G has an *XX* linkage minor, but that L is unique. Clearly having a vital linkage is preserved under taking linkage minors. But *XX* has two linkages, a contradiction. □

Next we show that the third statement of Theorem 1.2 implies the second.

Lemma 2.3. *For all n, $\ddot{U}_n$ has no XX linkage minor with respect to L_n.*

Proof. The result holds for $n \leq 2$, because then $|V(\ddot{U}_n)| < |V(XX)|$. Suppose the lemma fails for some $n \geq 3$, but is valid for all smaller n. Every edge of *XX* is incident with exactly one of the four end vertices of the paths. Hence all rung edges incident with at least two of the four end vertices are not in any *XX* linkage minor. But after deleting those edges from $\ddot{U}_n$ the end vertices have degree one, and hence the edges incident with them will not be in any *XX* linkage minor. Contracting these four edges produces $\ddot{U}_{n-2}$, a contradiction. □

Reversing a path P_i means exchanging the labels of vertices s_i and t_i, thereby reversing the order on the vertices of the path.

Lemma 2.4. *Let G be a graph, and L a chordless spanning linkage of order 2 of G consisting of paths P_1, running from s_1 to t_1, and P_2, running from s_2 to t_2. If G has no XX linkage minor, then G is a linkage minor of $\ddot{U}_n$ with respect to L_n for some integer n, such that L is a contraction of L_n.*

Proof. Suppose the statement is false. Let G be a counterexample with as few edges as possible. If some end vertex of a path, say s_1, has degree one (with $e = s_1 v$ the only edge), then we can embed G/e in $\ddot{U}_n$ for some n. Let G' be obtained from $\ddot{U}_n$ by adding four vertices s_1', t_1', s_2', t_2', and edges $s_1'v_1, s_1's_2', s_1't_2', s_2'u_1, s_2't_1', v_nt_1', u_nt_2', t_1't_2'$. Then G' is isomorphic to $\ddot{U}_{n+2}$, and G' certainly has G as linkage minor.

Hence we may assume that each end vertex of P_1 and P_2 has degree at least two. Suppose no rung edge runs between two of these end vertices. Then it is not hard to see that G has an *XX* minor, a contradiction. Therefore some two end vertices must be connected. By reversing paths as necessary, we may assume there is an edge $e = s_1 s_2$.

By our assumption, $G \backslash e$ can be embedded in $\ddot{U}_n$ for some n. Again, let G' be obtained from $\ddot{U}_n$ by adding four vertices s_1', t_1', s_2', t_2', and edges $s_1'v_1, s_1's_2', s_1't_2', s_2'u_1, s_2't_1', v_nt_1', u_nt_2', t_1't_2'$. Then G' is isomorphic to $\ddot{U}_{n+2}$, and G' certainly has G as linkage minor, a contradiction. □

As an aside, it is possible to prove a stronger version of the previous lemma. We say a partition (A, B) of the rung edges is *valid* if the edges in A are pairwise non-crossing, and the edges in B are pairwise non-crossing after reversing one of the paths. One can show:

- Each Truemper graph has a valid partition.
- For every valid partition (A, B) of a Truemper graph G, some $\ddot{U}_n$ has G as linkage minor in such a way that (A, B) extends to a valid partition of $\ddot{U}_n$.

Now we have all ingredients of our main result.

Proof of Theorem 1.2. From Lemma 2.2 we learn that (*i*)⟺(*ii*). From Lemma 2.3 we learn that (*iii*)⟹(*ii*), and from Lemma 2.4 we conclude that (*ii*)⟹(*iii*). □

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